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Optimal Mix of Passive and Active Control in Structures

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This paper shows how to redesign a structure to make it easier to control. We begin with any active controller (say, designed to give the "ideal" performance). Next, we divide this controller into two parts—one that can be synthesized simply by structure redesign (this we call the passive control) and one that is to be synthesized as an active controller. Given that the total closed-loop response must not be changed from the "ideal," the passive part of the controller is designed to minimize the amount of control power needed for the active part. Necessary and sufficient conditions for a globally optimum structure is given, and gradient calculations are not required. Two simple algorithms which converge to the global optimum solution are provided.

I. Introduction

IN the control of flexible structures, a structure is traditionally designed first, then a controller is designed for the given structure. This paper reverses this procedure, designing the structure after a controller is given. The idea is to start with the traditional wisdom, and characterize the dynamic behavior that the "ideal" controller would exhibit, working in closed loop with the given structure. But the controller is not synthesized in this way. This initial structure and controller design (which defines the "ideal" response) is only the first step (and this step is not the subject of this paper). The purpose of the paper is to show how to redesign the structure and the controller to make them work together, cooperating to achieve a common cause (the ideal dynamic response of the first step). This structure redesign is accomplished to reduce (minimize) the amount of active control power needed to achieve the ideal response. This is the key to the philosophy of our approach. We may say, therefore, that the structure is redesigned for better controllability.

To motivate these ideas consider that the structure and control design problems are not independent[15,18]; i.e., some structures are easier to control than others. Yet there exists no systematic integration of the disciplines of structure design and control design. Some valiant efforts to simultaneously optimize selected parameters of the structure and the controller appear in Refs. 1–10 and 17. The traditional structural optimization program seeks to minimize mass subject to side constraints on local stress or frequency of the first mode of vibration, etc. Even when the control parameters are simultaneously optimized, the types of controllers allowed are quite restricted (PID, etc.). Such problems are quite difficult and mathematically intractable (proofs of convergence are lacking), leading to computationally intensive, gradient-based algorithms, that can be ineffective for large-scale problems.

One might ask why should there be so much interest in integrating the structural and control design disciplines, after all, once a structure is designed (based upon a mature 150-year-old discipline), there are powerful control theories available, and separating the structure design and the control design in this way is easier for the analyst. Several answers to this question should be mentioned to motivate the paper.

First, the sophisticated theories of structure and control design have not been integrated to yield the best overall design. Today, the structure and controls designs are often competing, rather than helping each other. For example, robustness may be improved by using less control power. That is, a structure that is harder to control requires much more control power, but a large amount of control power, even slightly misdirected (due to modeling errors in the control design), can more easily destabilize the closed-loop system than a small amount of misdirected control energy. So when all the design resources (structure, control) are used wisely, more robust controllers result, without degrading nominal performance. That is, smarter structure designs might lead to more robust controllers, starting a genuinely beneficial merger of the disciplines. It is important to point out the disadvantage in relying entirely on robust control theory to take care of unwise structure designs and models. One usually must compromise nominal performance in order to gain robustness.[18]

Second, it is a more efficient use of resources to require that the structural dynamics and control dynamics cooperate with each other. In the past we have often taken to space our Earth-bound attitudes about structure design (where "larger mass," "the stiffer the better," and "the more damping the better" attitudes dominated our thinking), rather than allowing the absence of gravity and the presence of active control to suggest totally new ways of holding a system together. For example, a minimum mass (very flexible) structure might be put into space only to be controlled as a rigid body (e.g., Skylab, Space Station, and Hubble), because the convincing proof of an integrated approach was not available. The literature is replete with controllers designed to slew a flexible space body from one position to another, with part of the controller trying to keep elastic deflections small or zero, and the other part of the controller trying to follow a slew maneuver that would be optimal for a rigid body. But, more control energy is needed to slew the large moment of inertia (associated with the body held rigid) that is, ironically, forced upon the system by the regulator part of the controller that is fighting elastic motion. Here, control energy is used to combat the natural structural dynamics as opposed to allowing the natural structural dynamics and the controller dynamics to cooperate for the common good. Other examples of one component fighting another is the "one-at-a-time" closing of control loops. The potential advantage of this approach is the simplicity of using single-input, single-output control design techniques, avoiding the more difficult design which simultaneously coordinates all components. One component of a system can be "fighting" another if the dynamics of both components are not utilized in the design of the other. That is, we cannot wisely design components of a system one-at-a-time. The disadvantage of the one-at-a-time approach is that the structure components dynamics and the controller components have not been designed

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to cooperate to achieve an efficient overall design. Quite a few spacecraft control designs have followed the one-at-a-time philosophy (e.g., Hubble and Skylab), but the price we were willing to pay yesterday for such convenience is worth mentioning for pedagogical reasons. Using one-at-a-time loop design for the three-axis attitude control of Skylab, the controller used 90% of the attitude control power to fight the dynamic coupling between axes, leaving only 10% to accomplish the real objectives of attitude control. The loss of a solar panel further exaggerated the importance of control power inefficiencies. Because of the order of magnitude increase in control power (beyond what would be required by allowing the dynamics of the spacecraft to cooperate with the controller design) such a controller was also less robust to certain kinds of parameter variations.

A focus on integrating component designs (the structure design and the controller design) is also needed to keep the gulf between the disciplines from growing even wider. The popular misconception that increasing damping will improve the performance and robustness of controlled space structures is a good example of such widening. This belief has even led to a concentrated effort to maximize the damping in a space structure, and has led some to write specifications for control design in terms of a specified amount of increase in the damping factor of specified modes of vibration of the structure. The low authority-high authority controllers[20] also use these ideas to design a rate feedback loop (component one) to increase the damping prior to design of the outer position control loop (component two). But, the outer loop sometimes struggles to "undo" what the inner (rate) loop has done, wasting valuable control energy. We wish to show in this paper that sometimes the damping should be increased and sometimes decreased, depending on the structure to be controlled and the dynamic response objectives.

It is also a popular misconception that stiffening the structure will help the control objectives. Hence, the propensity to increase stiffness and damping makes such structures much more massive than they might need to be for efficient control. Our examples show that less control effort is often required by making the structure softer. This is diametrically opposed to the classical notion preferring to control rigid bodies, struggling eloquently to avoid exciting the structural modes. It is not at all clear that more robust designs are achieved in this way, by sending more power to the actuators (to force the flexible system to behave as a rigid body) as opposed to using the less power that would result by designing the structure to cooperate with the controller. This latter quest is the focus of this research: to design the structure to require the least control effort. Since smaller masses held with softer springs are easier to push around, the required control effort tends to decrease with mass. Hence, the algorithms in Sec. IV tend to (but not always) reduce the structural mass. This synergism allows the structure design and the controller design to be mutually beneficial, rather than becoming the victims of negative design attitudes suggested by such common goals to minimize "control structure interaction," "control and observation spillover," etc. For a given structure design and a given set of dynamic response requirements, it takes more control power to avoid spillover than would be required by making the structural dynamics help the controller dynamics. The irony here is that those of us in the control structures interation (CSI) community may find that controllers which control (cooperate with) flexibility may be more robust than controllers which avoid (the bandwidth of) flexibility.

Finally, an integrated approach might become an "enabling" technology for missions with stringent performance requirements that cannot be achieved using the separated theories of structures and control design.

We begin with a given output feedback controller, which has already been designed to satisfy performance requirements. This can be any feedback controller such as those based upon H_∞, LQ, pole assignment, etc. It is considered to be the controller that yields the ideal response, and the required choices of sensor and actuator locations to allow such a design have already been made. However, the given active controller will not be synthesized as an active controller. The controller will be synthesized in two parts, which we shall call the active and the passive parts. The active part is synthesized as a feedback controller using electronic sensors and actuators mounted on the structure. The passive part is synthesized by changing the physical parameters of the structure (such as cross-sectional areas of members, mass, stiffness of elements, etc.). The structure is redesigned (this is the passive control) to minimize the active part of the control effort. This two-step procedure has some advantages over "simultaneous control and structure optimization"[1-10,17]; since the necessary conditions are much easier to solve, performance (multiple output rms limits, etc.) is guaranteed by the design, and the minimum active control power is determined which will give the same response as the original given active controller. This means, for example, that if the design objectives can be met with only passive control (structure redesign), then our procedure will yield a zero active part of the controller. The procedure minimizes the feedback control that a structure needs. This is an important feature which seems to be introduced here for the first time.

II. Formulation of the Problem

Consider the following structural dynamics system with output feedback control:

$$M\ddot{q} + D\dot{q} + Kq = Bu \tag{1}$$

$$u = Gz \tag{2}$$

$$z = H\begin{pmatrix} q \\ \dot{q} \end{pmatrix} \tag{3}$$

where

$$q(t) \in \mathcal{R}^{n_q}, \qquad u(t) \in \mathcal{R}^{n_u}, \qquad z(t) \in \mathcal{R}^{n_z}$$

$$M = M^T > 0, \qquad K = K^T \geq 0, \qquad D = D^T \geq 0$$

We assume that G is given; i.e., the control law, Eq. (2), has already been designed such that the closed-loop system has acceptable performance (such as satisfaction of certain constraints on the dynamic response, e.g., from OVC, IVC controllers,[11] or assignment of a certain covariance matrix to the closed-loop system, e.g. covariance controllers,[12] or H_∞, or pole assignment).

Structure redesign (we shall call this passive control) can also be used to improve the response by any of the cited criteria,[13] but the use of only passive control cannot usually deliver the same level of performance as active control. The structure redesign concept is to change the mass matrix by ΔM, the damping matrix by ΔD, and the stiffness matrix by ΔK to yield the new system

$$(M+\Delta M)\ddot{q} + (D+\Delta D)\dot{q} + (K+\Delta K)q = Bu_a \tag{4}$$

$$u_a = G_a z \tag{5}$$

$$z = H\begin{pmatrix} q \\ \dot{q} \end{pmatrix} \tag{6}$$

satisfying the objectives, where G_a is the (active) part of the controller after redesign.

The principal idea is to separate the active control law, Eq. (2), into a passive part which is implemented into the physical system by redesign and an active part which constitutes the remaining active control law required after structure redesign. Therefore, the control law is written into the following form:

$$Bu = BGH\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = BG_aH\begin{pmatrix} q \\ \dot{q} \end{pmatrix} - \Delta M\ddot{q} - \Delta D\dot{q} - \Delta Kq \quad (7)$$

and the closed-loop system after redesign is

$$(M+\Delta M)\ddot{q} + (D+\Delta D)\dot{q} + (K+\Delta K)q = BG_aH\begin{pmatrix} q \\ \dot{q} \end{pmatrix} \quad (8)$$

where

$$u_a = G_aH\begin{bmatrix} q \\ \dot{q} \end{bmatrix} \quad (9)$$

is the active part of the controller and $\Delta M\ddot{q} + \Delta D\dot{q} + \Delta Kq$ is the passive part. The structure redesign criterion is to minimize the control power (a suitable norm of u_a) needed to satisfy Eq. (7) for any given G.

Note that the closed-loop system response before and after redesign remains unchanged, therefore all of the designed closed-loop system properties (e.g., disturbance rejection, closed-loop pole location, covariance matrix, etc.) remain unchanged. Simply put, the structure will be redesigned to help the controller achieve these objectives. Our objective is to find the passive control (ΔM, ΔD, ΔK) to minimize the active control U = norm of G_aHx, where $x^* = (q^* \dot{q}^*)$, where the controller G is given.

Let B_k, B_d, and B_m be the spring, stiffness, and mass connectivity matrices of the structural system. Therefore, any changes in the structural parameters k_i to $k_i + \Delta k_i$, d_i to $\Delta d_i + \Delta d_i$, and m_i to $m_i + \Delta m_i$ can be expressed in the form[13]:

$$K + \Delta K = K + B_kG_kB_k^T \quad (10)$$

$$D + \Delta D = D + B_dG_dB_d^T \quad (11)$$

$$M + \Delta M = M + B_mG_mB_m^T \quad (12)$$

where

$$G_k = \text{diag}(\ldots \Delta k_i \ldots) \quad (13)$$

$$G_d = \text{diag}(\ldots \Delta d_i \ldots) \quad (14)$$

$$G_m = \text{diag}(\ldots \Delta m_i \ldots) \quad (15)$$

This yields the following representation for the desired control law, Eq. (7):

$$BGH\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = BG_aH\begin{pmatrix} q \\ \dot{q} \end{pmatrix} - [\Delta K \quad \Delta D]\begin{pmatrix} q \\ \dot{q} \end{pmatrix} - \Delta M\ddot{q}$$

Substituting the solution of $\ddot{q}$ from Eqs. (1-3) yields

$$BGHx = (G_{\text{active}} + G_{\text{passive}})x, \qquad x^T = (q^T, \dot{q}^T) \quad (16)$$

Using Eqs. (10-15), we can write the control gains in Eq. (16) in the following compact form:

$$G_{\text{active}} = BG_aH \quad (17a)$$

$$G_{\text{passive}} = -I_oB_pG_pB_p^TL \quad (17b)$$

where

$$B_p = \begin{bmatrix} B_k & 0 & 0 \\ 0 & B_d & 0 \\ 0 & 0 & B_m \end{bmatrix} \quad (18)$$

$$I_o = [I \quad I \quad I] \quad (19)$$

$$G_p = \begin{bmatrix} G_k & 0 & 0 \\ 0 & G_d & 0 \\ 0 & 0 & G_m \end{bmatrix} \quad (20)$$

and

$$L = \begin{bmatrix} I \\ M^{-1}(BGH - [K \quad D]) \end{bmatrix} \quad (21)$$

In the following, we will assume that the input matrix B has full column rank and that the output matrix H has full row rank (independent controls and sensors).

The necessary and sufficient condition to enable us to decompose the control law into an active and a passive part as in Eq. (16), is given as follows.

Lemma 1: There exists an active controller G_a to satisfy

$$BGH = BG_aH - I_oB_pG_pB_p^TL \quad (22)$$

if and only if

$$BB^+I_oB_pG_pB_p^TLH^+H = I_oB_pG_pB_p^TL \quad (23)$$

and if this condition is satisfied, G_a is given by

$$G_a = G + B^+I_oB_pG_pB_p^TLH^+ \quad (24)$$

where $(\ )^+$ denotes the Moore-Penrose inverse of a matrix.

Remark: There always exists a G_p which solves Eq. (23) (for example, $G_p = 0$).

This is a well-known result from linear algebra,[14] and the fact that B and H have full column and full row rank, respectively, so that $B^+B = I$ and $HH^+ = I$.

For the passive control law to be physically realizable, it must satisfy certain inequality constraints imposed by the physics of the problem. For example, the stiffness and damping of any element of the system after redesign cannot be negative, and the mass of any element cannot decrease lower than a specific bound. [We should be careful here not to impose more constraints than is necessary. For example, it might be acceptable to allow some masses and members to go to zero (connections that are not needed should be removed); on the other hand, the mass of a scientific instrument (e.g., mirror) should not be allowed to go to zero in order to preserve the mission function.] Let

$$S = \begin{bmatrix} \ddots & & & & 0 \\ & k_i & & & \\ & & \ddots & & \\ & & & d_i & \\ & & & & \ddots \\ & & & & & m_i \\ 0 & & & & & & \ddots \end{bmatrix} = \text{diag}[\ldots s_i \ldots]$$

then these constraints can be represented as

$$G_p + S \geq C \quad (25)$$

where $C = \text{diag}[\ldots c_i \ldots]$ is the matrix with diagonal elements containing the specified lower bound values of the structural elements after redesign.

The system can be put into the state space form as follows:

$$\dot{x} = Ax + Bu \quad (26)$$

$$z = Hx \tag{27}$$

$$u = Gz \tag{28}$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix} \tag{29}$$

We shall assume that the system Eq. (26) is excited only through initial conditions (even though one can write an equivalent pulse input disturbance[15]). It is convenient to define the time correlation of the state

$$X \triangleq \sum_{i=1}^{nx} \int_0^\infty x(i,t)x^T(i,t)\,dt$$

(sometimes called the deterministic "covariance" matrix) where $x(i,t)$, $[u_a(i,t)]$ denotes $x(t)$ $[u_a(t)]$ when only the ith initial condition $x_i(0)$ is applied. The solution of the following Lyapunov equation yields X (Ref. 15):

$$(A + BGH)X + X(A + BGH)^T + X_0 = 0$$

$$X_0 = \mathrm{diag}[\ldots x_i^2(0)\ldots] \tag{30}$$

We seek to minimize the "power" of the active part of control law given by

$$U \triangleq \sum_{i=1}^{nx} \int u_a^T(i,t)Ru_a(i,t)\,dt = \mathrm{trace}(G_aHXH^TG_a^TR)$$

where R is a positive definite weighting matrix. Now, formulation of the problem for minimizing the active control power can be written as follows:

$$\underset{G_P}{\text{minimize}}\ \mathrm{trace}\,(G_aHXH^TG_a^TR) \tag{31}$$

subject to Eq. (24)

$$G_a = G + B^+I_oB_pG_pB_p^TLH^+ \tag{32}$$

Eq. (23)

$$BB^+I_oB_pG_pB_p^TLH^+H = I_oB_pG_pB_p^TL \tag{33}$$

and Eq. (25)

$$G_p + S \geq C \tag{34}$$

III. Necessary and Sufficient Conditions for Minimum Active Control Power

In this section, we will use the following notation. For any square $n \times n$ matrix $A = (a_{ij})$, define

$$\mathrm{diag}(A) = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & \\ \vdots & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}, \quad \mathrm{vec\,diag}(A) = \begin{bmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{nn} \end{bmatrix}$$

The Hadamard (or term by term) product of two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ is defined as

$$A \circ B = (a_{ij}b_{ij}) \tag{35}$$

The following two results can easily be proved.

Lemma 2: Let A, B, C, and D be square matrices where D is diagonal, $C = C^T$, and the products ADC and $ADCDB$ are defined. Then

$$\frac{\partial \mathrm{trace}(ADB)}{\partial D} = \mathrm{diag}(A^TB^T) = \mathrm{diag}(BA)$$

$$\frac{\partial \mathrm{trace}(ADCDB)}{\partial D} = \mathrm{diag}(A^TB^TDC + BADC)$$

where $(\ )^T$ denotes matrix transposition.

Lemma 3: Let A, B, and C be square matrices where B is diagonal, $C = C^T$, and the product ABC is defined. Then

$$\mathrm{vec\,diag}(ABC) = (A \circ C)\,\mathrm{vec\,diag}(B) \tag{36}$$

Now we can state the following result.

Theorem 1: If G_p minimizes Eq. (31) subject to the constraints Eqs. (32–34), then there exists matrices N and $\Lambda = \mathrm{diag}(\ldots\lambda_i\ldots)$ such that

$$BB^+I_oB_pG_pB_p^TLH^+H = I_oB_pG_pB_p^TL \tag{37}$$

$$\Lambda \geq 0 \tag{38}$$

$$\Lambda(G_p + S - C) = 0 \tag{39}$$

$$-(G_p + S - C) \leq 0 \tag{40}$$

$$P\,\mathrm{vec\,diag}(G_p) = r \tag{41}$$

where

$$P = (B_p^TI_o^TB^{+T}RB^+I_oB_p) \circ (B_p^TL^TH^+HXH^+HLB_p) \tag{42}$$

$$r = \mathrm{vec\,diag}(\Lambda - B_p^TI_o^TB^{+T}RGHXH^+HL^TB_p + B_p^TI_o^TN^TL^TB_p - B_p^TI_o^TBB^+N^TH^+HL^TB_p) \tag{43}$$

Proof: The Lagrangian of the problem is

$$\mathcal{L} = \mathrm{trace}(G_aHXH^TG_a^TR) + \mathrm{trace}N[BB^+I_oB_pG_pB_p^TLH^+H - I_oB_pG_pB_p^TL] + \mathrm{trace}\Lambda[-(G_p + S - C)]$$

Therefore substituting Eq. (32) for G_a and using Lemma 2, the first-order necessary conditions for the minimum are

$$\frac{1}{2}\frac{\partial \mathcal{L}}{\partial G_p} = \mathrm{diag}(B_p^TI_o^TB^{+T}RB^+I_oB_pG_pB_p^TLH^+HXH^+HL^TB_p + B_p^TI_o^TB^{+T}RGHXH^+HL^TB_p + B_p^TI_o^TB^{+T}B^TN^TH^+HL^TB_p - B_p^TI_o^TN^TLB_p - \Lambda) = 0 \tag{44a}$$

$$\frac{\partial \mathcal{L}}{\partial N} = BB^+I_oB_pG_pB_p^TLH^+H - I_oB_pG_pB_p^TL = 0 \tag{44b}$$

$$-(G_p + S - C) \leq 0 \tag{45a}$$

$$\Lambda(G_p + S - C) = 0 \tag{45b}$$

$$\Lambda \geq 0 \tag{45c}$$

So using Lemma 2, we can derive the necessary condition Eq. (44). Considering a Taylor's series expansion of trace

$G_a HXH^T G_a^T R$, the second-order term in the expansion

$$\text{trace}(B^+ I_o B_p \Delta G_p B_p^T LH^+ HXH^+ HL^T B_p \Delta G_p B_p^T I_o^T B^{+T} R) \geq 0 \tag{46}$$

is obviously non-negative for all feasible ΔG_p. □

Remark: Equations (44a) and (44b) are linear in the unknowns G_p and N. Hence, iteration is required only due to inequality constraints.

Because the function being minimized is quadratic, subject to linear inequality constraints, the necessary conditions developed are also sufficient, and moreover, the problem has a global minimum solution. Since the second derivative Eq. (46) is only positive semidefinite, the global minimum might not be unique.

IV. Methodology for Solution of the Optimization Problem

Next, we propose two simple algorithms to numerically solve the optimization problem. The first algorithm corresponds to a variation of the Lagrange multipliers method and the second is a penalty method.

The first proposed algorithm is the following.

Structure Redesign Algorithm 1

1) Initialize:

$$\lambda_i(0) = \frac{1}{s_i}$$

2) Solve the linear algebra problem Eq. (44) for N and G_p (we only need G_p).

3) Update λ_i as follows:

If $|G_{pii}| \leq s_i - c_i$ set: $\lambda_i(n+1) = \lambda_i(n)\left(\frac{|G_{pii}|}{s_i - c_i}\right)^\alpha$

If $G_{pii} > s_i - c_i$ set: $\lambda_i(n+1) = \lambda_i(n)\left[\frac{s_i - c_i}{G_{pii}}\right]^\alpha$

If $G_{pii} < c_i - s_i$ set: $\lambda_i(n+1) = \lambda_i(n)\left[-\frac{G_{pii}}{s_i - c_i}\right]^\alpha$

Where α is given number, return to step 2, unless for all i and some selected small ϵ

$$|G_{pii}(n+1) - G_{pii}(n)| < \epsilon$$

The second approach to solve numerically the constrained optimization problem is to use a penalty approach, i.e., to replace the problem by an unconstrained minimization problem of form

minimize

$$\text{trace}(G_a HXH^T G_a^T R) + \rho\,\text{trace}\left[Z(G_p + S - C)^2\right] + \rho\,\text{trace}\{(BB^+ I_o B_p G_p B_p^T LH^+ H - I_o B_p G_p B_p^T L)(BB^+ I_o B_p G_p B_p^T LH^+ H - I_o B_p G_p B_p^T L)^T\} \tag{47}$$

where $Z = \text{diag}(\ldots z_i \ldots)$ and ρ a positive penalty scalar which is allowed to go to infinity. The constant scalars z_i are chosen such that $z_i = 1$ if the corresponding inequality constraint $G_{pii} + s_i - c_i \geq 0$ is active, and $z_i = 0$ if the constraint is non-active. Minimization of Eq. (47) requires that the following first-order necessary condition is satisfied:

$$P_1 \,\text{vec diag}(G_p) = r_1 \tag{48}$$

where

$$P_1 = (B_p^T I_o^T B^{+T} R B^+ I_o B_p) \circ (B_p LH^+ HXH^+ HL^T B_p) + \rho(B_p^T I_o^T I_o B_p) \circ (B_p^T LH^+ HL^T B_p) - \rho(B_p^T I_o^T BB^+ I_o B_p) \circ (B_p^T LH^+ HL^T B_p) + Z \tag{49}$$

and

$$r_1 = -\text{vec diag}(B_p^T I_o^T B^{+T} RGHXH^+ HL^T B_p) - \rho Z\,\text{vec diag}(S - C) \tag{50}$$

Therefore, the following algorithm can be used to find the optimal solution, where it is assumed that the matrix P_1 is invertible. This algorithm is based on an active constraint set strategy.[16]

Structure Redesign Algorithm 2

1) Initialize: $z_i(0) = 0$

2) $G_p = P_1^{-1} r_1$.

3) Update z_i:

If $G_{pii} + s_i - c_i \geq 0$ set $z_i(n) = 0$

If $G_{pii} + s_i - c_i < 0$ set $z_i(n) = 1$

Return to step 2, unless for all i

$$G_{pii}(n+1) - G_{pii}(n) = 0$$

The problem should be solved for different values of the penalty parameters ρ, and the solution of the constrained optimization problem corresponds to the limit of the solutions when $\rho \to \infty$.

V. Numerical Examples

Consider the lumped mass, fully connected system (Fig. 1) where $m_1 = 4$, $m_2 = 2$, $m_3 = 10$; $k_1 = 2$, $k_2 = 2$, $k_3 = 4$, $k_4 = 1$, $k_5 = 3$, $k_6 = 1$; and $d_1 = 0.42$, $d_2 = 0.22$, $d_3 = 1.04$, $d_4 = 0.01$, $d_5 = 0.03$, $d_6 = 0.01$. The damping elements satisfy the Rayleigh damping assumption for the system. The system connectivity matrices are

$$B_k = B_d = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$B_m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the mass, spring, and damping matrices are

$$M = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix} \qquad K = \begin{bmatrix} 6 & -1 & -3 \\ -1 & 4 & -1 \\ -3 & -1 & 8 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.46 & -0.01 & -0.03 \\ -0.01 & 0.24 & -0.01 \\ -0.03 & -0.01 & 1.08 \end{bmatrix}$$

Let's consider the input and measurement matrices to be the following:

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad H = I$$

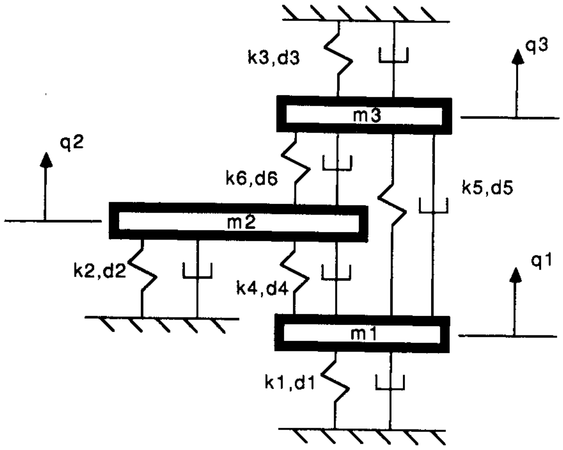

Fig. 1 Mechanical system of examples 1 and 2.

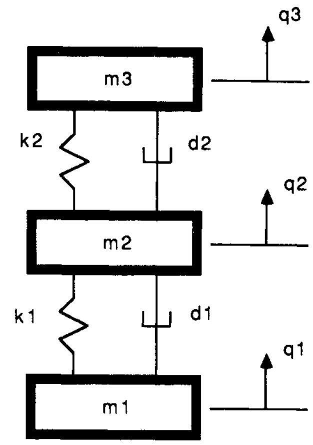

Fig. 2 Mechanical system of example 3.

Suppose that the following controller

$$G = \begin{bmatrix} 0.0766 & -0.1368 & -0.1888 & -0.0813 & -1.0269 & -0.1964 \\ 0.0845 & 0.0062 & -0.1701 & -0.0454 & -0.0393 & -1.2739 \end{bmatrix}$$

has been designed for this system to satisfy given requirements (assign a specified covariance matrix to the closed-loop system). Our structure redesign objective is then to minimize the active control power. The required control power before the redesign is (for $R = I$)

$$\text{trace}(GHXH^TG^T) = 0.2972$$

Example 1

Let the constraints on the physical elements of the system be such that the spring and dampers after redesign have non-negative values and the masses satisfy

$$m_1 \geq 0, \qquad m_2 \geq 0, \qquad m_3 \geq 5$$

So the constraint matrix C from Eq. (25) is

$$C = \text{diag}(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 5)$$

i.e., we allow the physical elements of the structure to go to zero, except the mass m_3 which is restricted to have a value greater than or equal to 5. Both algorithms presented in Sec. IV gave the following results:

$$G_p = \text{diag}(-0.9701 \quad -0.8903 \quad -1.9518 \quad -0.4850 \quad -1.4551$$
$$-0.5213 \quad -0.2037 \quad 0.4102 \quad 0.1038 \quad -0.0049$$
$$-0.0146 \quad -0.0100 \quad -1.9401 \quad -0.9848 \quad -5.000)$$

$$G_a = \begin{bmatrix} 0.0315 & 0.0035 & -0.0669 & -0.0414 & -0.0077 & -0.0946 \\ -0.0026 & 0.0244 & -0.0132 & -0.0231 & -0.0146 & -0.0177 \end{bmatrix}$$

$$U = \text{trace}(G_aHXH^TG_a^T) = 8.0273 \times 10^{-4}$$

Therefore, the optimal changes of the structural elements should be

$$\Delta k = [-0.9701 \quad -0.8903 \quad -1.9518 \quad -0.4850 \quad -1.4551 \quad -0.5213]$$

$$\Delta d = [-0.2037 \quad 0.4102 \quad 0.1038 \quad -0.0049 \quad -0.0146 \quad -0.0100]$$

$$\Delta m = [-1.9401 \quad -0.9848 \quad -5.000]$$

and the percentage reduction of the required active control power is

$$[(0.2972 - 8.0273 \times 10^{-4})/0.2972] \times 100 = 99.7\%$$

Note that the optimal solution corresponds to a reduction of the masses m_1, m_2, and m_3; but only the mass m_3 is achieving its minimum allowed value. The spring coefficients are also decreasing but not reaching their minimum allowed value zero. Some of the dampers are increasing and some are decreasing although only the damper d_6 is reaching its minimum value zero. Hence, the redesigned structure is softer and less massive than the original structure, and the problem is nontrivial (and therefore interesting) in the sense that masses and springs are not taken to the trivial value of zero.

Example 2

In this example, suppose we restrict the structural elements to satisfy

$$k_i \geq 0, \qquad d_i \geq 0, \qquad i = 1, \ldots, 6$$
$$m_1 \geq 4, \qquad m_2 \geq 2, \qquad m_3 \geq 10$$

i.e., we do not allow any mass reduction. The results in this case are

$$\Delta k = [0.0599 \quad 0.2194 \quad 0.0965 \quad 0.0299 \quad 0.0898 \quad -0.0426]$$

$$\Delta d = [0.0126 \quad 1.0405 \quad 1.2476 \quad 0.0003 \quad 0.0009 \quad -0.0100]$$

$$\Delta m = [0.1197 \quad 0.0304 \quad 0.0]$$

$$G_a = \begin{bmatrix} 0.0631 & 0.0070 & -0.1388 & 0.0827 & -0.0154 & -0.11896 \\ -0.0053 & 0.0488 & -0.0265 & -0.0463 & 0.0293 & -0.0354 \end{bmatrix}$$

$$U = \text{trace}(G_a H X H^T G_a^T) = 0.0032$$

The required input power is reduced by $[(0.2792-0.0032)/0.2792]\times 100 = 98.9\%$. The masses m_1 and m_2 are increasing and all dampers and springs are increasing except d_6 and k_6. The d_6 reaches its minimum allowed value zero, and k_6 results in a slight reduction.

Next, consider the system in Fig. 2, where $m_1 = 4$, $m_2 = 2$, $m_3 = 10$; $k_1 = 1$, $k_2 = 1$; and $d_2 = 0.01$, $d_3 = 0.01$, and where the matrices B and H are the same as in previous examples. The system connectivity matrices are

$$B_k = B_d = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \qquad B_m = I$$

Suppose the following linear quadratic optimal controller has been designed for this system:

$$G = \begin{bmatrix} 0.2432 & -0.5968 & -0.6344 & -1.6659 & -1.6264 & -1.0877 \\ -0.0637 & -0.1523 & -0.6930 & -0.2704 & -0.2175 & -2.8266 \end{bmatrix}$$

The required control power before redesign is

$$\text{trace}(GHXH^T G^T) = 0.9245$$

Example 3

Suppose the constraints at the structural elements are (similar to example 1) $k_i \geq 0$, $d_i \geq 0$, for $i = 1,2$; and $m_1 \geq 0$, $m_2 \geq 0$, $m_3 \geq 5$. The resulted optimal changes are

$$\Delta k = [0.0 \qquad -0.4910]$$

$$\Delta d = [0.0 \qquad -0.3550]$$

$$\Delta m = [0.0 \quad -0.9153 \quad -5.0000]$$

$$U = \text{trace}(G_a H X H^T G_a^T) = 0.4123$$

and the required input power is reduced by 55.4%. Note that the mass m_3 reaches its lower limit 5.

Example 4

Consider a two-bay portal frame shown in Fig. 3. The structure is assumed to be fixed at the base. The control objective is to keep the maximum displacement and velocity rms (root mean squared) values at node 4 below specified values. At node 2 there are thrust actuators in the x and y directions and a torquer (about the z axis). Displacements and velocities are measured at all nodes. The material properties for this structure are as follows: mass density $\rho = 2.591 \times 10^{-4}$ (lb$_f$-s^2/in.^{4}) and modulus of elasticity $E = 10^7$ (lb$_f$-in.$^{-2}$).

This frame moves only in the x-y plane and is modeled using combined beam-flexure finite elements and a consistent mass formulation. Each element has three degrees of freedom at each end. The structure redesign is accomplished by changing the cross-sectional dimension of each truss member b_i, $i = 1,2,\ldots,10$. The initial design has a square cross-sectional area 1 in.^{2}. The damping matrix is assumed proportional with a maximum damping ratio of 1%.

The outputs y_1 and y_2 represent displacements in the x and y directions, and the outputs y_3 and y_4 represent velocities in the x and y directions, all at node 4. An initial controller is designed to minimize the control power required to keep the rms values of each of the four outputs below σ_i^2, $i = 1,\ldots,4$, where

$$[\sigma_1^2 \sigma_2^2 \sigma_3^2 \sigma_4^2] = [0.005 \quad 0.00015 \quad 5000 \quad 1000]$$

The optimal controller using the OVC algorithm in Refs. 11, 15, and 19 solves this problem:

$$V_u = \text{trace}(GHXH^* G^* R) = 4.2584 \times 10^4$$

Now allow the depth b_i of each member of the structure to vary with a minimum value zero. The ith member is numbered in sequence corresponding to Fig. 3. Since the damping is small, we shall assume that it is not a function of the member areas.

After redesign, the new control power and the percent reduction becomes

$$\Delta b_1 = 0.7268 \text{ in.}$$

$$\Delta b_2, \Delta b_3 \ldots \Delta b_6 = 0 \text{ in.}$$

$$\Delta b_7 = -0.8022 \text{ in.}$$

$$\Delta b_8, \Delta b_9, \Delta b_{10} = 0 \text{ in.}$$

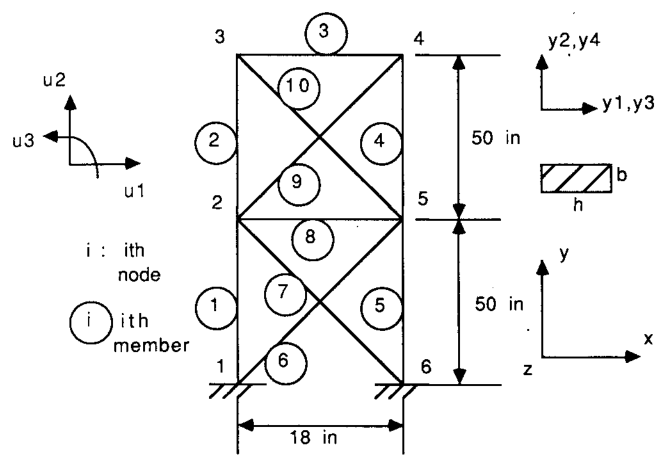

Fig. 3 Two-bay frame of example 4.

$$V_{u_a} = \text{trace}(G_a H X H^* G_a^* R) = 3.8721 \times 10^4$$

Hence, there is $[(4.2584 - 3.8721)/3.8721] = 9.07\%$ control power reduction. Note that the truss member 7 is virtually eliminated, and member 1 was stiffened by 72.68%. This has the effect of reducing the mass at the actuator locations and inertia and stiffness in the direction of the control forces. This is a logical way to reduce the control power needed. All other members remain unchanged. The structure redesign saves the active controller about 9% in control power.

VI. Conclusion

This paper presents a method to design structures for control so as to minimize the amount of active control needed after structure design. Traditionally, the structure is designed first and then a controller is designed for the given structure. This paper reverses this procedure and designs a structure for a given controller.

First, an output feedback controller C_0 is designed for an initial structure S_0. In this paper, this initial controller C_0 is assumed given. It matters not what the origins of C_0 are. (For example, it could be an H_∞ controller, an LQ controller, or one designed by pole-assignment or covariance assignment.) The combination of S_0 and C_0 yields the so-called "ideal" response.

The second step is to redesign the structure (changing physical parameters in the original structure) and the active controller so as to minimize the amount of active control power that will be needed after the structure redesign. The new structure S_1 and the new controller C_1 yield a closed-loop response which matches that of the original structure S_0 in closed loop with the original controller C_0.

Generally, one would not expect a unique set of masses, dampers, and springs that will give a mechanical system the same response. However, of all possible admissible ways to modify the structure to match the response, this paper finds the values for the masses, dampers, and springs (in lumped parameter models) that allow the control objectives to be met with the smallest possible amount of active control effort. This means that the active controller will be automatically eliminated by the design procedure if it is possible to satisfy the dynamic response objectives with structural changes alone. But, more importantly, the design procedure integrates structure and control design in a tractable way.

Most structural optimization problems have in the past been stated as minimal mass problems, subject to side constraints. The possible advantages of our approach are threefold: 1) the side constraints we impose might be more realistic (constraints on changes in individual masses, dampers, springs), 2) the control law is not so restricted (any output feedback controller can be used to set the standards of performance to be matched), and 3) the globally optimal solution is obtained in a finite number of iterations. (This makes the method more useful for large-scale systems.)

The key to these attractive features of our results is the minimization of control power rather than a direct attempt to minimize mass. This turns out to yield an easier quadratic mathematical program since the necessary conditions are linear in the unknowns (and this allows guarantees of convergence and global optimality). Moreover, since it takes less power to push around smaller masses, we conjecture that the minimization of the control power is an indirect means to reduce mass. The algorithm also finds the optimal damping and stiffness for the structure, subject to the given performance constraints (matching the response of the ideal system).

From the given examples, we can conclude that the optimal structure redesign can provide a large improvement in the active control power required. In these examples, 99% reduction in control power was achieved for a certain fully connected system, 55% reduction was achieved for a less connected system, and 9% reduction was achieved in a two-bay truss redesign. The optimal solution is sometimes obtained by increasing and sometimes by decreasing the mass, stiffness, and damping of the structural system, depending on the constraints imposed on the physical elements. An important but not surprising conclusion is that (for a given performance requirement) usually (in our examples) softer, more flexible structures require less control power than more rigid structures. This contradicts the notion that controls ought to be designed to make the structure behave as a rigid body. Hence, an expected benefit of advancing control structures interaction (CSI) techniques is that an integrated approach requires less control by intentionally controlling flexibility. The quantification of robustness will be investigated in future work.

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